Math 3310: Exam 2

1. Let $(G, \times)$ be a group and H a nonempty subset of G. Prove that $(H, \times)$ is a subgroup of G if and only if for all $a, b \in H$, we have $ab^{-1} \in H$.

2. Let X be any set and let $(G, \cdot)$ be a group. Let $hom_{set}(X, G)$ be the set of all functions with domain X and codomain G. Prove that $(hom_{set}(X, G), *)$, where $(f * g)(x) := f(x) \cdot g(x)$, is a group.

3. The *power set* of a set A, denoted $\mathcal{P}(A)$, is defined as the set of all subsets of A. For example, if $A = \{a, b\}$, $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Let X and Y be sets. Prove that, for any function $f : X \to Y$, there is an induced function $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ such that, if $g : Y \to Z$, then $\mathcal{P}(g \circ f) = \mathcal{P}(g) \circ \mathcal{P}(f)$, where $\mathcal{P}(g \circ f) : \mathcal{P}(X) \to \mathcal{P}(Z)$.

4. Let G be a group. Prove that there exists a group F such that $\#hom_{grp}(F, G) = 1$, where $hom_{grp}(F, G)$ is the set of all homomorphisms from F to G.